

# EUCLIDEAN MINIMAL TORI WITH PLANAR ENDS AND ELLIPTIC SOLITONS

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**ABSTRACT.** A Euclidean minimal torus with planar ends gives rise to an immersed Willmore torus in the conformal 3-sphere  $S^3 = \mathbb{R}^3 \cup \{\infty\}$ . The class of Willmore tori obtained this way is given a spectral theoretic characterization as the class of Willmore tori with reducible spectral curve. A spectral curve of this type is necessarily the double of the spectral curve of an elliptic KP soliton. The simplest possible examples of minimal tori with planar ends are related to 1-gap Lamé potentials, the simplest non-trivial algebro geometric KdV potentials. If one allows for translational periods, Riemann's "staircase" minimal surfaces appear as other examples related to 1-gap Lamé potentials.

## 1. INTRODUCTION

Complete minimal surfaces with finite total curvature and planar ends in Euclidean 3-space can be compactified by filling in points at the ends if one views them as immersions into the conformal 3-sphere  $S^3 = \mathbb{R}^3 \cup \{\infty\}$ . This is equivalent to compactifying their preimage under stereographic projection. The main interest in this Möbius geometric compactification stems from the fact that the resulting immersions of compact surfaces are critical points of the Willmore energy. In fact, as proven by Bryant [9, 10], all Willmore spheres in the conformal 3-sphere can be obtained from this construction. This is not anymore true for Willmore immersions of genus  $g \geq 1$ . For example, the Clifford torus has Willmore energy  $W = 2\pi^2$ , while compactifications of Euclidean minimal surfaces with planar ends always have Willmore energy  $W = 4\pi n$  for  $n$  the number of ends.

In the present paper we characterize Willmore tori in conformal 3-space  $S^3$  that are Euclidean minimal with planar ends for some point  $\infty \in S^3$  at infinity in terms of spectral and integrable systems theory. Our integrable systems approach leads to a simple description of the Euclidean minimal tori previously studied by Costa [12] and Kusner, Schmitt [19] which have  $W = 16\pi$  and four ends, the least number of ends possible for Euclidean minimal tori with planar ends. It equally applies to minimal tori with planar ends and translational periods like Riemann's "staircase" minimal surfaces, see e.g. [16, 20, 21, 22]. From the spectral theory point of view all these examples turn out to be related to the simplest non-trivial algebro geometric KdV potentials, the 1-gap Lamé potentials. This observation also sheds new light on the recent characterization of Riemann's minimal surfaces obtained by Meeks, Perez and Ros [21, 22] which uses algebro geometric solutions to the KdV equation in an essential way.

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The way that spectral and integrable systems theory intervenes in our setting is through the *spectral curve* of conformally immersed tori. The spectral curve is an invariant of tori immersed into 3-dimensional space first considered in [27]. It is defined as the Riemann surface normalizing the Floquet spectrum of a 2-dimensional Dirac operator

$$(1.1) \quad D = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & \bar{U} \end{pmatrix}.$$

attached to an immersion. The spectral curve is asymptotic to the “vacuum” spectrum belonging to the operator with zero potential  $U = 0$  which is the disjoint union of two copies of  $\mathbb{C}$  with a  $\mathbb{Z}^2$ -lattice of double points. For a generic immersion the spectral curve has infinite genus, because infinitely many of the vacuum double points turn into handles. In the special case that only finitely many handles appear in the asymptotics, the spectral curve has finite genus and the immersion can be constructed using finite gap integration. This happens for example, if the immersion is the solution to elliptic variational problems related to the area [24, 17] or Willmore functional [7].

The fact that all previously known examples of spectral curves occurring in surface theory were irreducible made people expect that irreducibility should hold for general conformally immersed tori (see e.g. the “Pretheorem” in [28]). In the present paper we show that this is not true and prove (see Theorem 2.3 below):

**Theorem.** *Every Euclidean minimal torus  $f: T^2 \rightarrow \mathbb{R}^3 \cup \{\infty\}$  with planar ends has a reducible spectral curve.*

By general spectral curve theory (cf. Appendix A.4), reducibility of the spectral curve implies in particular that the spectral curve has two irreducible components of finite genus which are interchanged under an anti-holomorphic involution. Moreover, each component is a compact Riemann surface with one puncture. As a consequence, each component of the spectral curve is the spectral curve of an elliptic KP soliton in the sense of [18] (cf. Appendix B). This makes contact to the theory of elliptic Calogero–Moser systems [1, 18].

Combining our result with the fact that a Willmore torus in the conformal 3-sphere  $S^3$  that is not Euclidean minimal with planar ends has an irreducible spectral curve of finite genus (see Theorem 5.1 and Corollary 5.3 of [7]), we obtain the following reformulation of the above theorem:

**Theorem.** *Every immersed Willmore torus  $f: T^2 \rightarrow S^3$  in the conformal 3-sphere  $S^3$  has finite spectral genus. The spectral curve of  $f$  is reducible if and only if  $f$  is Euclidean minimal with planar ends with respect to the Euclidean geometry defined by some point  $\infty \in S^3$  at infinity.*

To our knowledge, Dirac operators (1.1) corresponding to Euclidean minimal tori with planar ends are the first known examples of 2-dimensional periodic Dirac operators with smooth potential and reducible Floquet spectral curve. (Such Dirac potentials are indeed globally smooth, because Euclidean minimal tori with planar ends extend to globally smooth immersions into the conformal 3-sphere  $S^3 = \mathbb{R}^3 \cup \{\infty\}$ .)

In the following we sketch our argument why Euclidean minimal tori with planar ends have reducible spectral curves. A conformally immersed torus  $f: T^2 \rightarrow S^3$  gives rise to Dirac type operators with smooth potentials in two different ways. One is Möbius invariant (Section A.2), the other one depends on the Euclidean geometry defined by the choice of a point  $\infty \in S^3$  at infinity *not lying on the image of the immersion* (Section A.3).

By Theorem A.6, all different Dirac type operators obtained this way from a conformally immersed torus give rise to the same spectral curve, the spectral curve of the immersion  $f$ .

A general point  $\infty \in S^3$  at infinity that *lies on the image of the immersion* gives rise (as in Section A.3) to a Dirac type operator with a non-smooth potential. However, in the special case that the resulting immersion into Euclidean 3-space is minimal (and then necessarily has planar ends), the corresponding Dirac operator has a trivial potential, i.e., is of the form (1.1) with  $U = 0$ .

The main result of the paper (Theorem 2.3) is derived from the fact that the spectral curve of a Euclidean minimal torus with planar ends can be computed using this Dirac operator (1.1) with trivial potential  $U = 0$  if one imposes suitable boundary conditions at the ends. The reducibility of the spectral curve then reflects the decomposition of the Dirac operator into pure  $\bar{\partial}$ - and  $\partial$ -operators. In particular, the components of the spectral curve coincide (Corollary 2.5) with spectral curves of  $\bar{\partial}$ -operators on punctured elliptic curves as discussed in [8]

## 2. THE SPECTRAL CURVE OF A EUCLIDEAN MINIMAL TORUS WITH PLANAR ENDS

In the proof of our main result we make use of the quaternionic approach to conformal surface geometry, see Appendix A, and treat minimal tori with planar ends in Euclidean 3-space as conformal immersions  $f: T^2 \rightarrow \mathbb{HP}^1$  with values in the conformal 3-sphere  $\mathbb{R}^3 \cup \{\infty\} \subset \mathbb{HP}^1$ , where  $\mathbb{R}^3 = \text{Im } \mathbb{H}$  is identified with  $\{[x, 1] \mid x \in \text{Im } \mathbb{H}\} \subset \mathbb{HP}^1$  and  $\infty = [1, 0]$ .

We call an immersion  $f: M \rightarrow \mathbb{R}^3 \cup \{\infty\}$  of a compact surface  $M$  *Euclidean minimal with planar ends* if it is globally smooth and the immersion  $f: M \setminus \{p_1, \dots, p_n\} \rightarrow \mathbb{R}^3$  is Euclidean minimal, where  $\{p_1, \dots, p_n\}$  denotes the finite set of *ends* at which  $f$  goes through  $\infty$ . The minimal immersion  $f: M \setminus \{p_1, \dots, p_n\} \rightarrow \mathbb{R}^3$  is then complete, has finite total curvature, and planar ends and, conversely, every such immersion can be compactified to an immersion into the conformal 3-sphere, see [9, pp. 44–49] which moreover proves:

**Lemma 2.1.** *A map  $f: M \rightarrow \mathbb{R}^3 \cup \{\infty\}$  defined on a compact Riemann surface  $M$  is conformal and Euclidean minimal with planar ends  $\{p_1, \dots, p_n\}$  if and only if the  $\mathbb{C}^3$ -valued 1-form  $\partial f$  is null<sup>1</sup>, holomorphic, and nowhere vanishing on  $M \setminus \{p_1, \dots, p_n\}$  and has second order poles without residues at the  $\{p_1, \dots, p_n\}$ .*

We describe now how the spinorial Weierstrass representation of minimal surfaces in  $\mathbb{R}^3$  appears in the quaternionic framework. Recall that for an immersion  $f: M \rightarrow \mathbb{R}^3 \cup \{\infty\}$  into the conformal 3-sphere, the corresponding quaternionic line subbundle  $L \subset V$  of the trivial rank 2 bundle  $V$  carries a Möbius invariant structure of a quaternionic spin bundle with compatible  $\bar{\partial}$ -operator (Appendix A.3). The fact that the immersion  $f$  is Euclidean minimal with planar ends is equivalent to the fact that the Euclidean quaternionic holomorphic structure induced by  $\infty$  (Appendix A.3) has vanishing Hopf field  $Q$  and coincides with the underlying  $\bar{\partial}$ -operator (the reason being that the Hopf field of a Euclidean holomorphic line bundle coincides with the mean curvature half density, see e.g. [23], and hence vanishes precisely if the immersion is minimal). Note that, unlike the

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<sup>1</sup>Here  $\partial f := \frac{1}{2}(df - i * df)$ , where  $*$  denotes the induced complex structure on  $T^*M$  and  $i$  stands for the complex structure of the complexification  $\mathbb{C}^3$  of  $\mathbb{R}^3$ ; the fact that  $\partial f$  is null with respect to the complex bilinear extension of the Euclidean metric reflects conformality of the immersion.

Möbius invariant underlying  $\bar{\partial}$ -operator, the quaternionic holomorphic section  $\psi$  appearing in the Weierstrass representation is only defined away from the ends  $\{p_1, \dots, p_n\}$  at which the immersion goes through  $\infty$ .

In the case that  $M = T^2 = \mathbb{C}/\Gamma$  is a torus, the canonical bundle is holomorphically trivialized  $K \cong \mathbb{C}$  by the differential  $dz$  of a uniformizing coordinate. The complex spin bundle  $E$  underlying  $L$  thus has a nowhere vanishing  $\bar{\partial}$ -holomorphic section  $\varphi$  with  $\mathbb{Z}_2$ -monodromy  $h_0 \in \text{Hom}(\Gamma, \mathbb{Z}_2)$  satisfying  $(\varphi, \varphi) = jdz$ . The quaternionic holomorphic section  $\psi$  appearing in the Weierstrass representation (Appendix A.3) then takes the form  $\psi = \varphi(s_1 + is_2j)$ , where  $s_1, s_2$  are holomorphic functions with  $\mathbb{Z}_2$ -monodromy  $h_0$  defined away from the ends. Hence

$$(2.1) \quad df = (\psi, \psi) = (js_1 + is_2)dz(s_1 + is_2j) = j(s_1^2 dz - \bar{s}_2^2 d\bar{z}) + 2i \text{Re}(s_1 s_2 dz)$$

and, with respect to the basis  $i, j, k$  of  $\mathbb{R}^3 = \text{Im } \mathbb{H}$ , we obtain the spinorial<sup>2</sup> Weierstrass representation

$$(2.2) \quad df = \text{Re} \begin{pmatrix} 2s_1 s_2 dz \\ (s_1^2 - s_2^2) dz \\ i(s_1^2 + s_2^2) dz \end{pmatrix}$$

for minimal surfaces, see [30] for the original and [3, 19] for contemporary, coordinate independent versions. Because  $\partial f$  in Lemma 2.1 coincides (up to a factor 1/2) with the  $\mathbb{C}^3$ -valued 1-form in (2.2), we obtain:

**Corollary 2.2.** *A map  $f: T^2 \setminus \{p_1, \dots, p_n\} \rightarrow \mathbb{R}^3$  is minimal with planar ends  $p_1, \dots, p_n$  if and only if the meromorphic functions  $s_1, s_2$  in (2.2) have poles of order at most one at the ends  $p_1, \dots, p_n$ , vanishing order zero terms in their Laurent expansions at the ends, and no common zeroes.*

Conversely, if one starts with meromorphic functions  $s_1, s_2$  on  $T^2 = \mathbb{C}/\Gamma$  with identical  $\mathbb{Z}_2$ -monodromy and zero and pole behavior as in the corollary, then (2.2) defines the differential of a possibly double periodic minimal torus with planar ends. The vanishing of the translational periods in direction of  $\gamma \in \Gamma$  is then equivalent to

$$(2.3) \quad \int_{\gamma} s_1 s_2 dz \in i\mathbb{R} \quad \text{and} \quad \int_{\gamma} s_1^2 dz = \int_{\gamma} \bar{s}_2^2 d\bar{z}.$$

In order to obtain a closed torus this has to hold for all  $\gamma \in \Gamma$ ; for a torus with one translational period it has to hold for one generator of  $\Gamma$ .

**Theorem 2.3.** *The spectral curve of a Euclidean minimal torus  $f: T^2 \rightarrow \mathbb{R}^3 \cup \{\infty\}$  with planar ends is reducible.*

Following from general properties of spectral curves, cf. Appendix A.4, the spectral curve of  $f$  is thus of the form  $\Sigma = \Sigma' \cup \bar{\Sigma}'$  for  $\Sigma'$  a Riemann surface of finite genus with one end. More precisely  $\Sigma'$  is the spectral curve belonging to an elliptic KP soliton, see Appendix B. Theorem 2.3 holds more generally for Euclidean minimal tori with planar ends and translational monodromy like Riemann's "staircase" minimal surfaces, see Section 4.

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<sup>2</sup>If  $M$  is not a torus, but an arbitrary Riemann surface, the argument of this paragraph only holds locally; if  $K$  is not globally trivialized, a global formula can be obtained by absorbing  $\sqrt{dz}$  and the  $\mathbb{Z}_2$ -monodromy into  $s_1, s_2$  and viewing them as spinor fields instead of functions (then  $dz$  disappears in (2.2) and  $\varphi$  further above has to be divided by  $\sqrt{dz}$ ), see e.g. [3, 19] for this globalized version.

*Proof.* The basic idea behind the proof is similar to that of Theorem A.6. An essential difference is that here we chose a point  $\infty$  at infinity for which the immersion has ends, i.e., goes through  $\infty$ . As a consequence, the corresponding flat connection  $\nabla$  on  $V/L$  with  $\nabla\psi_1 = 0$  (cf. Appendix A.3) is only defined away from the ends, because the section  $\psi_1$  vanishes at the ends. Our strategy in order to determine the spectral curve of  $V/L$  is to derive a characterization of sections of  $KV/L \cong L$  that are defined away from the ends and are of the form  $\nabla\psi^h$  for  $\psi^h$  a holomorphic section of  $V/L$  with monodromy  $h$ .

As above denote by  $\varphi$  a non-trivial  $\bar{\partial}$ -holomorphic section with  $\mathbb{Z}_2$ -monodromy  $h_0$  of the complex line bundle  $E$  underlying  $L$ . Let  $\psi^h$  be a holomorphic section of  $V/L$  with monodromy  $h$ . It can be written as  $\psi^h = \psi_1\chi$ , with  $\chi$  a quaternion valued function with monodromy  $h$  defined away from the ends. Using the identification  $KV/L \cong L$  via  $\delta$ , its derivative is then

$$\nabla\psi^h = \psi_1 d\chi \cong \varphi(\Phi_1 + \Phi_2 j),$$

where  $\Phi_1, \Phi_2$  are complex holomorphic functions with monodromy  $h \cdot h_0$  and  $\bar{h} \cdot h_0$  defined away from the ends (that  $\Phi_1, \Phi_2$  are complex holomorphic holds, because  $\nabla\psi^h$  is a holomorphic section of  $KV/L \cong L$ , cf. Appendix A.3, whose Hopf field  $Q$  vanishes identically for  $f$  Euclidean minimal). From  $\nabla\psi_2 = -\psi_1 df \cong -\varphi(s_1 + is_2 j)$  we thus obtain

$$(2.4) \quad d\chi = df(s_1 + is_2 j)^{-1}(\Phi_1 + \Phi_2 j) = (js_1 + is_2)dz(\Phi_1 + \Phi_2 j),$$

where the last equality holds by (2.1).

We prove now that  $\Phi_1$  and  $\Phi_2$  obtained from a holomorphic section  $\psi^h$  with monodromy of  $V/L$  have the same pole behavior as  $s_1$  and  $s_2$ , i.e., their only possible poles are first order poles at the ends and their Laurent expansions at the ends have vanishing order zero terms. To prove the first claim, note that near every end the holomorphic section  $\psi_2 = -\psi_1 f$  (Appendix A.2) is non-vanishing so that  $\psi^h = \psi_2 \chi_2$  for a smooth quaternion valued function  $\chi_2$ . Moreover, because  $\psi_1 = -\psi_2 f^{-1}$  is holomorphic and  $df^{-1}$  is non-zero at a planar end of  $f$ , there is a smooth quaternionic function  $g_2$  defined by  $d\chi_2 = -df^{-1}g_2$  (this is in fact the quotient construction of Appendix A.2). Taking the derivative of  $\chi = -f\chi_2$  yields

$$(2.5) \quad d\chi = -df(\chi_2 + f^{-1}g_2) = -dfg_1 \quad \text{with} \quad g_1 := \chi_2 + f^{-1}g_2.$$

Because  $g_1$  is smooth and, by (2.4), away from the ends

$$(2.6) \quad g_1 = -(s_1 + is_2 j)^{-1}(\Phi_1 + \Phi_2 j),$$

the pole behavior of  $s_1$  and  $s_2$  implies that  $\Phi_1$  and  $\Phi_2$  at most have first order poles at the ends.

The fact that  $d\chi$  in (2.4) has no periods around a given end is equivalent to the fact that the closed forms

$$js_1\bar{\Phi}_1 dz + is_2\bar{\Phi}_2 dzj \quad \text{and} \quad is_2\bar{\Phi}_1 dz - \bar{s}_1\bar{\Phi}_2 d\bar{z}$$

have vanishing residues at that end (i.e., the integrals  $\frac{1}{2\pi i} \oint$  over small loops around the end are zero). Because the Laurent expansions of  $s_1$  and  $s_2$  have no order zero terms at the end, the order zero terms  $\Phi_1(0)$  and  $\Phi_2(0)$  in the Laurent expansions of  $\Phi_1$  and  $\Phi_2$  have to satisfy

$$\begin{pmatrix} s_1(-1) & i\bar{s}_2(-1) \\ is_2(-1) & \bar{s}_1(-1) \end{pmatrix} \begin{pmatrix} \Phi_1(0) \\ \bar{\Phi}_2(0) \end{pmatrix} = 0,$$

where  $s_1(-1)$  and  $s_2(-1)$  denote the residues of  $s_1$  and  $s_2$  at the end. Because at least one of  $s_1$  and  $s_2$  has a pole at the end, the determinant  $|s_1(-1)|^2 + |s_2(-1)|^2$  of this  $2 \times 2$  matrix is non-zero and both  $\Phi_1, \Phi_2$  have vanishing order zero terms at the end.

Thus, every holomorphic section  $\psi^h$  of  $V/L$  with monodromy  $h$  can be obtained, by integrating (2.4), from meromorphic functions  $\Phi_1$  and  $\Phi_2$  with monodromies  $h \cdot h_0$  and  $\bar{h} \cdot h_0$  and first order poles with vanishing order zero terms at the ends.

We prove now that conversely every pair of meromorphic functions  $\Phi_1$  and  $\Phi_2$  with the given properties comes from a holomorphic section  $\psi^h$  of  $V/L$  with monodromy  $h$ . By Lemma 2.6 below, the form  $d\chi$  obtained by plugging  $\Phi_1$  and  $\Phi_2$  into (2.4) can be integrated to a smooth function  $\chi$  with monodromy  $h$  which is defined away from the ends of the immersion. It therefore remains to check that the holomorphic section  $\psi^h = \psi_1\chi = \psi_2\chi_2$  thus defined smoothly extends through the ends. By (2.6), the asymptotics of  $s_1, s_2$  and  $\Phi_1, \Phi_2$  at the ends implies that the function  $g_1$  is smooth and  $dg_1$  vanishes at the ends. Using (2.5) and the defining equation of  $g_2$ , the differential of  $g_1$  becomes  $dg_1 = f^{-1}dg_2$ . Hence  $dg_2 = fdg_1$  is bounded at the ends and smooth elsewhere so that by integration  $g_2$  is  $C^0$  at the ends. But now  $d\chi_2 = -df^{-1}g_2$  implies that  $\chi_2$  is  $C^1$  at the ends. The holomorphic section  $\psi_1\chi = \psi_2\chi_2$  is thus  $C^1$  at the ends and hence smooth by elliptic regularity.

So far we have shown that holomorphic sections  $\psi^h$  of  $V/L$  with monodromy  $h$  correspond to pairs of meromorphic functions  $\Phi_1$  and  $\Phi_2$  with monodromies  $h \cdot h_0$  and  $\bar{h} \cdot h_0$  and prescribed pole behaviors. Because for generic points of the spectral curve the corresponding space of holomorphic sections of  $V/L$  with monodromy  $h$  is complex 1-dimensional (Appendix A.4), generically either  $\Phi_1$  or  $\Phi_2$  has to vanish identically. (Otherwise they would give rise to a complex 2-dimensional space of holomorphic sections with monodromy  $h$ , because one could separately plug  $\Phi_1$  and 0 or 0 and  $\Phi_2$  into (2.4).)

This shows that the spectral curve has two connected components, namely one on which the corresponding holomorphic sections  $\psi^h$  generically have vanishing  $\Phi_2$  and one on which  $\Phi_1$  vanishes generically. Both parts are interchanged under the anti-holomorphic involution which is induced by the symmetry  $\psi^h \mapsto \psi^{\bar{h}}$ .  $\square$

In the proof of Theorem 2.3 we have derived the following characterization of multipliers  $h$  admitting non-trivial holomorphic sections of  $V/L$  with monodromy  $h$ :

**Lemma 2.4.** *Let  $f: T^2 \rightarrow \mathbb{R}^3 \cup \{\infty\}$  be a Euclidean minimal torus with planar ends at  $p_1, \dots, p_n \in T^2$  and induced spin structure corresponding to a  $\mathbb{Z}_2$ -multiplier  $h_0$ . Then  $V/L$  admits a holomorphic section  $\psi^h$  with monodromy  $h$  if and only if there is a meromorphic function  $\Phi$  with monodromy  $h \cdot h_0$  or  $\bar{h} \cdot h_0$  that has at most first order poles and vanishing order zero terms at the  $p_1, \dots, p_n \in T^2$ .*

Because the spectral curve is defined as the Riemann surface normalizing the set of Floquet multipliers  $h$  that belong to non-trivial holomorphic sections of  $V/L$ , it can be computed by the following corollary.

**Corollary 2.5.** *The set of multipliers  $h \in \text{Hom}(\Gamma, \mathbb{C}_*)$  admitting a non-trivial meromorphic function  $\Phi$  with monodromy  $h$  on  $T^2 = \mathbb{C}/\Gamma$  such that*

- a) *all poles are of first order and located at the ends  $p_1, \dots, p_n \in T^2$  and*
- b) *the Laurent series at the ends have vanishing order zero terms*



is a 1-dimensional complex analytic set. Its normalization is one connected component of the spectral curve of the minimal torus with planar ends and hence a compact Riemann surface with one puncture.

In Theorem 2.3 and the following discussion we derive properties of spectral curves (like number of ends and components and finiteness of the genus) of Euclidean minimal tori with planar ends from the general theory of spectral curves of immersed tori in the conformal 3-sphere (Appendix A.5), a special case of the theory of spectral curves for periodic 2-dimensional Dirac operators (Appendix A.4).

Corollary 2.5 indicates an alternative way to study spectral curves of Euclidean minimal tori with planar ends. This is further discussed in [8], where we more generally define spectral curves of  $\bar{\partial}$ -operators on punctured elliptic curves with boundary conditions as described in Corollary 2.5 and show that they coincide with elliptic KP spectral curves. It turns out that Krichever's ansatz [18] yields an algebraic approach to spectral curves and Floquet functions of  $\bar{\partial}$ -operators on punctured elliptic curves with the given boundary conditions.

**Appendix to Section 2.** The following lemma is needed in the proof of Theorem 2.3.

**Lemma 2.6.** *Let  $\omega$  be a meromorphic 1-form with non-trivial monodromy  $h \in \text{Hom}(\Gamma, \mathbb{C}_*)$  and vanishing residues on a torus  $T^2 = \mathbb{C}/\Gamma$ . Then there exists a unique meromorphic function  $f$  with monodromy  $h$  on  $T^2$  that satisfies  $df = \omega$ .*

*Proof.* Take  $\tilde{f}$  an arbitrary meromorphic function with  $d\tilde{f} = \omega$ . Because  $\omega_{z+\gamma} = \omega_z h_\gamma$  for  $\gamma \in \Gamma$ , there is  $a_\gamma \in \mathbb{C}$  such that  $\tilde{f}(z+\gamma) - \tilde{f}(z)h_\gamma = a_\gamma$  for all  $z$ . Now

$$\tilde{f}(z + \gamma_1 + \gamma_2) = \tilde{f}(z)h_{\gamma_1}h_{\gamma_2} + a_{\gamma_1}h_{\gamma_2} + a_{\gamma_2}$$

and, because  $\tilde{f}(z + \gamma_1 + \gamma_2) = \tilde{f}(z + \gamma_2 + \gamma_1)$  and  $h$  is a representation, we obtain  $a_{\gamma_1}h_{\gamma_2} + a_{\gamma_2} = a_{\gamma_2}h_{\gamma_1} + a_{\gamma_1}$  and hence

$$(2.7) \quad a_{\gamma_1}(h_{\gamma_2} - 1) = a_{\gamma_2}(h_{\gamma_1} - 1).$$

In particular, because  $h$  is non-trivial, we have  $a_\gamma = 0$  for all  $\gamma \in \Gamma$  such that  $h_\gamma = 1$ . On the other hand, adding a constant  $b \in \mathbb{C}$  to  $\tilde{f}$  changes  $a_\gamma$  to  $a_\gamma + b(1 - h_\gamma)$ . For  $\gamma \in \Gamma$  such that  $h_\gamma \neq 1$  we define  $b = \frac{a_\gamma}{h_\gamma - 1}$ . By (2.7) the definition of  $b$  does not depend on the choice of  $\gamma$  and  $b$  is the unique solution to  $a_\gamma + b(1 - h_\gamma) = 0$  for all  $\gamma \in \Gamma$ . In particular, among the meromorphic functions satisfying  $df = \omega$ , the function  $f = \tilde{f} + b$  is the unique one with multiplicative monodromy  $h$  (and no additional “additive monodromy”).  $\square$

### 3. EXAMPLE: MINIMAL TORI WITH FOUR PLANAR ENDS AT THE HALF PERIODS

The smallest number of ends possible for Euclidean minimal tori with planar ends is four, as shown by Kusner and Schmitt [19]. This is analogous to the case of minimal spheres with planar ends [9] which, except for the plane, have also at least four ends. The first examples of Euclidean minimal tori with four planar ends are given by Costa [12] (who treats rectangular tori) and Kusner and Schmitt [19] and have ends located at the half periods.

In the following we show that the spectral curve of a minimal torus with four planar ends located at the half periods is the double of the spectral curve of a 1-gap Lamé potential. As we will see, it is natural to view the parameter domain of the Euclidean minimal torus

as a 4-fold covering of the spectral curve belonging to the Lamé potential; the four ends of the minimal surface then cover the one end of the Lamé spectral curve. The spectral curve point of view yields natural candidates for the spinors fields  $s_1$  and  $s_2$  needed to construct the minimal immersion.

**Theorem 3.1.** *The spectral curve of a Euclidean minimal torus with four planar ends located at the half periods is the double  $\Sigma = \Sigma' \dot{\cup} \bar{\Sigma}'$  of the spectral curve  $\Sigma'$  of a 1-gap Lamé potential.*

*Proof.* By Corollary 2.5, one can compute one component  $\Sigma'$  of the spectral curve  $\Sigma = \Sigma' \dot{\cup} \bar{\Sigma}'$  by normalizing the 1-dimensional set of multipliers  $h \in \text{Hom}(\Gamma, \mathbb{C}_*)$  for which there exists a meromorphic function  $\Phi$  with monodromy  $h$ , at most first order poles at the ends, and vanishing order zero term in the Laurent expansion at each end. The normalization of the set of possible  $h$ 's is an irreducible compact Riemann surface with one puncture.

Denote by  $T^2 = \mathbb{C}/\tilde{\Gamma}$  with  $\tilde{\Gamma} = \text{Span}_{\mathbb{Z}}\{4\omega_1, 4\omega_3\}$  a uniformization of the torus. Then  $\Sigma' = (\mathbb{C}/\Gamma) \setminus \{0\}$  with  $\Gamma = \frac{1}{2}\tilde{\Gamma} = \text{Span}_{\mathbb{Z}}\{2\omega_1, 2\omega_3\}$  is in a natural way a component of the spectral curve: for every  $\alpha \in \Sigma'$ , the 1-gap Lamé Baker–Akhiezer function  $\Phi_\alpha$  of  $\mathbb{C}/\Gamma$ , when seen as a function on the 4-fold covering  $T^2 = \mathbb{C}/\tilde{\Gamma}$  of  $\mathbb{C}/\Gamma$ , has exactly the right kind of Laurent expansion at the ends (see Appendix C.2). The monodromy of  $\Phi_\alpha$  on  $\mathbb{C}/\Gamma$  in the direction  $\gamma = 2\omega_j$  is  $e^{2(\zeta(\alpha)\omega_j - \alpha\eta_j)}$  so that its monodromy on the 4-fold covering  $T^2$  in the direction  $\tilde{\gamma} = 4\omega_j$  is  $h^{\tilde{\gamma}} = e^{4(\zeta(\alpha)\omega_j - \alpha\eta_j)}$ . Thus,  $\Sigma'$  is indeed a Riemann surface with one end that parametrizes a subset of the monodromies which, as in Corollary 2.5, are possible for meromorphic functions on  $T^2$  with first order poles and vanishing order zero terms at the half periods.  $\square$

In the following we explain how to reconstruct all Euclidean minimal tori with four planar ends at the half periods of  $T^2 = \mathbb{C}/\tilde{\Gamma}$  and spin structure corresponding to a given  $\mathbb{Z}_2$ -multiplier  $h_0 \in \text{Hom}(\tilde{\Gamma}, \mathbb{Z}_2)$ . For this one has to understand how to solve

- a) the *algebraic problem* of finding a two linearly independent meromorphic functions  $s_1, s_2$  with monodromy  $h_0$  as in Corollary 2.2 (i.e., with poles of order at most one at the half periods, with vanishing order zero terms in the Laurent expansions at the half periods, and without common zeroes),
- b) and the *period problem* (2.3).

One can check<sup>3</sup> that the algebraic problem cannot be solved for non-trivial spin structure  $h_0 \not\equiv 1$ . In the case of trivial spin structure  $h_0 \equiv 1$ , there is a 3-dimensional space of meromorphic functions on  $T^2 = \mathbb{C}/\tilde{\Gamma}$  whose only poles are first order poles at the half periods and whose Laurent expansions at the half periods have vanishing order zero terms. It is spanned by the Baker–Akhiezer functions  $\Phi_1 = \Phi_{\alpha=\omega_1}$ ,  $\Phi_2 = \Phi_{\alpha=\omega_2}$  and  $\Phi_3 = \Phi_{\alpha=\omega_3}$  on  $\mathbb{C}/\Gamma$  (see Appendix C.2) viewed as functions on the 4-fold covering  $T^2 = \mathbb{C}/\tilde{\Gamma}$ , for  $\Gamma = \text{Span}_{\mathbb{Z}}\{2\omega_1, 2\omega_3\}$  and  $\tilde{\Gamma} = \text{Span}_{\mathbb{Z}}\{4\omega_1, 4\omega_3\}$ .

Because the natural  $\mathbb{R}_+ \text{SU}(2)$ -action on  $s_1, s_2$  changes the immersion by a homothety only, the general ansatz for  $s_1$  and  $s_2$  of Euclidean minimal tori with four ends at the half

<sup>3</sup>In fact, for given  $h_0 \not\equiv 1$  the 4-dimensional space of meromorphic functions on  $T^2/\tilde{\Gamma}$  with monodromy  $h_0$  and first order poles at the half periods is spanned by translates of Baker functions  $\Phi_\alpha$  on  $T^2/\tilde{\Gamma}$ , cf. Appendix C.2, for  $\alpha$  equal to one of the  $\omega_i$ ,  $i = 1, \dots, 3$  depending on  $h_0$ ; but no linear combination of these translates has vanishing order zero terms in the Laurent expansion at all four half periods.



periods and trivial spin structure is

$$(3.1) \quad s_1 = \Phi_1 + a \Phi_2 + b \Phi_3$$

$$(3.2) \quad s_2 = c \Phi_2 + d \Phi_3$$

(that  $\Phi_1$  has a non-trivial coefficient can be achieved by renumbering the basis of  $\Gamma$  if necessary).

For the computation of the periods we use that

$$(3.3) \quad \begin{aligned} \Phi_k(x)\Phi_l(x) &= \tilde{\wp}(x) \pm \tilde{\wp}(x - 2\omega_1) \pm \tilde{\wp}(x - 2\omega_2) \pm \tilde{\wp}(x - 2\omega_3) \quad k \neq l \\ \Phi_k^2(x) &= \tilde{\wp}(x) + \tilde{\wp}(x - 2\omega_1) + \tilde{\wp}(x - 2\omega_2) + \tilde{\wp}(x - 2\omega_3) - \wp(\omega_k), \end{aligned}$$

where  $\wp$  and  $\tilde{\wp}$  denote the Weierstrass  $\wp$ -functions of  $\mathbb{C}/\Gamma$  and  $\mathbb{C}/\tilde{\Gamma}$ , respectively, and where in the first equation always two of the  $\pm$  are  $--$  signs (e.g. for  $\Phi_1\Phi_2$  we have  $-, -, +$ ). As a consequence  $\Phi_k\Phi_l dx$  with  $k \neq l$  has no periods at all and

$$(3.4) \quad \int_{\gamma=4\omega_l} \Phi_k^2 dx = -4(\eta_l + \wp(\omega_k)\omega_l) = -4(\eta_l + e_k\omega_l).$$

Evaluation (2.3) along  $\gamma = 4\omega_1$  and  $\gamma = 4\omega_3$  yields

$$(3.5) \quad \begin{aligned} \operatorname{Re}(ac(\eta_1 + e_2\omega_1) + bd(\eta_1 + e_3\omega_1)) &= 0 \\ \operatorname{Re}(ac(\eta_3 + e_2\omega_3) + bd(\eta_3 + e_3\omega_3)) &= 0 \end{aligned}$$

and

$$(3.6) \quad \begin{aligned} \overline{(\eta_1 + e_1\omega_1) + a^2(\eta_1 + e_2\omega_1) + b^2(\eta_1 + e_3\omega_1)} &= c^2(\eta_1 + e_2\omega_1) + d^2(\eta_1 + e_3\omega_1) \\ \overline{(\eta_3 + e_1\omega_3) + a^2(\eta_3 + e_2\omega_3) + b^2(\eta_3 + e_3\omega_3)} &= c^2(\eta_3 + e_2\omega_3) + d^2(\eta_3 + e_3\omega_3). \end{aligned}$$

A dimension count suggests that there is 2 real parameter space of solutions.

Given  $a, b, c, d$  such that (3.5) and (3.6) are solved, one still has to check that  $s_1$  and  $s_2$  have no common zeros (so that one obtains an immersion). The minimal torus with four planar ends is then given by plugging  $s_1$  and  $s_2$  into (2.2) and integrating. Because by (3.3) one only has to integrate  $\wp$ -functions on the parameter torus  $T^2$ , this yields an explicit formula of the minimal surface in terms of Weierstrass  $\zeta$ -functions on  $T^2$ .

*Example:* setting  $a = b = 0$ , (3.5) is solved automatically and by matrix inversion (3.6) gives  $c$  and  $d$  uniquely up to sign, because

$$(3.7) \quad \det \begin{pmatrix} \eta_1 + e_2\omega_1 & \eta_1 + e_3\omega_1 \\ \eta_3 + e_2\omega_3 & \eta_3 + e_3\omega_3 \end{pmatrix} = (e_3 - e_2) \det \begin{pmatrix} \eta_1 & \omega_1 \\ \eta_3 & \omega_3 \end{pmatrix} = (e_3 - e_2) \frac{\pi i}{2}.$$

The fact that the spinors  $s_1, s_2$  generically have no common zeros is checked in Section 23 of [19].

Examples of Euclidean minimal tori with more planar ends are constructed in [25]. It would be interesting to generalize our construction of tori with four ends to an explicit construction, based on elliptic soliton theory, of all Euclidean minimal tori with planar ends.

Apart from our construction of minimal tori with four planar ends, in minimal surface theory Baker–Akhiezer functions previously appeared in Bobenko’s paper [3] on helicoids with handles. In both cases the spinors describing the minimal surface are Baker–Akhiezer functions, so that the Euclidean minimal surfaces in question are parametrized by (coverings of) their own spectral curves. This relation between Euclidean minimal surfaces and integrable systems is fundamentally different from the well established theory [24, 17, 2]

of constant mean curvature tori in  $\mathbb{R}^3$  with  $H \neq 0$  (where the immersion is parametrized by a suitable real torus in the Jacobian of the spectral curve).

#### 4. SURFACES WITH TRANSLATIONAL PERIODS AND RIEMANN'S MINIMAL SURFACES

Euclidean minimal tori with planar ends and translational periods can be treated along the same lines as closed minimal tori with planar ends. In the following we investigate the simplest non-trivial<sup>4</sup> examples which are tori with one translational period and two planar ends on each fundamental domain. The embedded examples among these surfaces are Riemann's "staircase" minimal surfaces, see [20]. Like in Section 3, the spectral curves of Euclidean minimal tori with translational periods and two planar ends are doubles of 1-gap Lamé spectral curves.

For a Euclidean minimal torus with planar ends and translational periods, both the Möbius invariant quaternionic holomorphic line bundle  $V/L$  (Appendix A.2) and the Euclidean holomorphic line bundles  $L$  and  $KL^{-1}$  (Appendix A.3) are well defined. The main difference to the case with trivial translational monodromy is that the flat bundle  $V$  is no longer trivial. Moreover, the holomorphic section  $\psi_2$  is not periodic (instead  $\psi_1$  and  $\psi_2$  span a two-dimensional linear system with monodromy; the monodromies are  $2 \times 2$  Jordan blocks with 1 on the diagonal and translational periods appearing in the upper right corners). The holomorphic section  $\psi_1$  of  $V/L$  that corresponds to  $\infty$  is still periodic, because  $\infty$  is fixed under translations. Also, the holomorphic section  $\psi$  appearing in the Weierstrass representation is well defined. In particular, the functions  $s_1, s_2$  are well defined meromorphic functions with  $\mathbb{Z}_2$ -monodromy and pole and zero behavior as in Corollary 2.2. Only the periodicity condition (2.3) is not satisfied anymore.

The spectral curve of a minimal torus with planar ends and translational monodromy can still be defined as the spectral curve of the quaternionic holomorphic line bundle  $V/L$ . In particular, the proof of Theorem 2.3 goes through without change (the non-periodic holomorphic section  $\psi_2$  appears in the proof only in local considerations about the pole behavior of  $\Phi_1$  and  $\Phi_2$  and the smoothness of  $\psi^h = \psi_1\chi$ ).

**Lemma 4.1.** *A Euclidean minimal torus with two planar ends and translational periods has non-trivial spin structure and ends at two of the four half periods of the torus.*

*Proof.* Let  $T^2 = \mathbb{C}/\Gamma$  with  $\Gamma = \text{Span}_{\mathbb{Z}}\{2\omega_1, 2\omega_3\}$  and assume that the ends are located at 0 and  $p \in T^2$ . In order to see that the spin structure is not trivial, note that a meromorphic function with trivial monodromy and first order poles at 0 and  $p \in T^2$  is of the form

$$x \in T^2 \mapsto a(\zeta(x) - \zeta(x - p)) + b$$

with  $a, b \in \mathbb{C}$ . The condition that the order zero term in the Laurent expansion at both ends vanishes is  $a\zeta(p) + b = 0$ . It is thus impossible to find two linearly independent functions  $s_1$  and  $s_2$  as in Corollary 2.2 with trivial monodromy.

Therefore, the spin structure has to be non-trivial and we can assume that the corresponding  $\mathbb{Z}_2$ -multiplier  $h_0$  satisfies  $h_0(2\omega_2) = 1$  and  $h_0(2\omega_3) = -1$ . The two-dimensional space of meromorphic functions with monodromy  $h_0$  and first order poles at 0 and  $p$  is then spanned by the Baker functions  $x \mapsto \Phi_{\alpha=\omega_2}(x)$  and  $x \mapsto \Phi_{\alpha=\omega_2}(x - p)$  on  $\mathbb{C}/\Gamma$  (see Appendix C.2). The condition of Corollary 2.2 that the order zero terms in the Laurent

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<sup>4</sup>The trivial example here being the plane which can be viewed as a minimal torus with two translational periods and no ends on its fundamental domain.

expansion at the ends vanishes for both sections is  $\Phi_{\alpha=\omega_2}(p) = \Phi_{\alpha=\omega_2}(-p) = 0$ . But this only holds if  $p = \omega_2 \in T^2$ .  $\square$

The proof of the following theorem is analogous to that of Theorem 3.1. The fundamental domain of the torus with two ends is now a double covering of the Lamé spectral curve; the two ends cover the one end of the spectral curve.

**Theorem 4.2.** *The spectral curve of a Euclidean minimal torus with two planar ends and translational periods is the double  $\Sigma = \Sigma' \dot{\cup} \bar{\Sigma}'$  of a 1-gap Lamé spectral curve  $\Sigma'$ .*

*Proof.* As in Theorem 3.1, we use Corollary 2.5 to compute one component of the spectral curve. Denote by  $T^2 = \mathbb{C}/\Gamma$  with  $\Gamma = \text{Span}_{\mathbb{Z}}\{2\omega_1, 2\omega_3\}$  a uniformization of the torus such that the ends are located at 0 and  $\omega_2 = \omega_1 + \omega_3$ . Then  $\Sigma' = (\mathbb{C}/\Gamma') \setminus \{0\}$  with  $\Gamma' = \text{Span}_{\mathbb{Z}}\{\omega_2, 2\omega_3\}$  is one component of the spectral curve, because for every  $\alpha \in \Sigma'$  the Baker–Akhiezer function  $\Phi_\alpha$  on  $\mathbb{C}/\Gamma'$  (see Appendix C.2), when seen as a function on the 2-fold covering  $T^2 = \mathbb{C}/\Gamma$  of  $\mathbb{C}/\Gamma'$ , has first order poles and Laurent expansions with vanishing order zero terms at the ends.  $\square$

In the rest of the section we sketch how to recover the classification [20] of Euclidean minimal tori with two parallel planar ends and one translational period. In particular, we explain how Riemann’s “staircase” minimal surfaces appear in our setting. The general ansatz for spinors  $s_1$  and  $s_2$  belonging to minimal tori with two planar ends that are parallel to the  $jk$ -plane and located at the points 0 and  $\omega_2$  of  $T^2 = \mathbb{C}/\Gamma$  with  $\Gamma = \text{Span}_{\mathbb{Z}}\{2\omega_1, 2\omega_3\}$  is (because as in the proof of Lemma 4.1 the monodromy of  $s_1$  and  $s_2$  has to be  $h_0$  with  $h_0(2\omega_2) = 1$  and  $h_0(2\omega_3) = -1$ )

$$s_1 = \Phi_1 + \Phi_2 \quad \text{and} \quad s_2 = a(\Phi_1 - \Phi_2)$$

with  $a \in \mathbb{C}$ , where  $\Phi_1, \Phi_2$  denote the Baker functions  $\Phi_{\alpha=\omega_2/2}$  and  $\Phi_{\alpha=\omega_2/2+\omega_3}$  on  $\mathbb{C}/\Gamma'$  (see Appendix C.2) with  $\Gamma' = \text{Span}_{\mathbb{Z}}\{\omega_2, 2\omega_3\}$  seen as functions on the double covering  $T^2 = \mathbb{C}/\Gamma$ . The spinors  $s_1, s_2$  have no common zeros. For integrating (2.2) and computing the periods we use that (by comparing poles and zeros)

$$(4.1) \quad \begin{aligned} \Phi_1(x)\Phi_2(x) &= \wp(x) - \wp(x - \omega_2) \\ \Phi_j^2(x) &= \wp(x) + \wp(x - \omega_2) - b_j, \quad j = 1, 2 \end{aligned}$$

with  $b_1 = 2\wp(\omega_2/2)$  and  $b_2 = 2\wp(\omega_2/2 + \omega_3)$ . Thus, as in Section 3, (2.2) can be explicitly integrated in terms of  $\zeta$ -functions. Because  $\Phi_1\Phi_2 dx$  has no periods, the closedness (2.3) of the minimal immersion described by  $s_1$  and  $s_2$  in the direction  $\gamma \in \Gamma$  reads

$$a \int_{\gamma} (\Phi_1^2(x) - \Phi_2^2(x)) dx \in i\mathbb{R} \quad \text{and} \quad \int_{\gamma} (\Phi_1^2(x) + \Phi_2^2(x)) dx = \overline{\bar{a}^2 \int_{\gamma} (\Phi_1^2(x) + \Phi_2^2(x)) dx}.$$

The period in the direction  $\gamma_{m,n} = 2(m\omega_1 + n\omega_3)$ ,  $m, n \in \mathbb{Z}$  vanishes if

$$(4.2) \quad 2a(b_2 - b_1)(m\omega_1 + n\omega_3) \in i\mathbb{R}$$

with

$$(4.3) \quad \bar{a} = \pm \arg(8(m\eta_1 + n\eta_3) - 2(b_1 + b_2)(m\omega_1 + n\omega_3))$$

for  $\arg(z) = z/|z|$ . This suggests that for given  $m, n$  there is a 1-parameter family of points in the moduli space of genus one curves for which the  $\gamma_{m,n}$ -period is closed, i.e., (4.2) is satisfied for  $a$  prescribed via (4.3) by  $m, n$ . A more detailed investigation of this would recover the classification given in Theorem 3.2 of [20].

The embedded examples in the moduli space of all Euclidean minimal tori with two parallel planar ends are Riemann's "staircase" minimal surfaces (see Theorem 3.1 of [20]). They are parametrized over rectangular tori and appear in our setting for  $\omega_1 \in \mathbb{R}$  and  $\omega_3 \in i\mathbb{R}$ . Because this implies  $b_2 = \bar{b}_1$ , for  $\gamma_{m,n}$  with  $mn = 0$  and the corresponding choice (4.3) of  $a$ , equation (4.2) is then automatically satisfied. The parametrizations of Riemann's minimal surfaces thus obtained essentially coincide with the ones given in [16].

Riemann's minimal surfaces have previously been characterized using algebro geometric KdV theory in the work of Meeks, Perez, and Ros [21, 22]. It would be interesting to conceptually relate the way KdV theory appears in their work with our approach based on spectral curve theory.

## APPENDIX A. SOME NOTIONS OF QUATERNIONIC HOLOMORPHIC GEOMETRY

We review the relevant notions of quaternionic holomorphic geometry [23, 13, 11]. In particular we discuss the quaternionic holomorphic approach [5, 4] to the spectral curve of conformally immersed tori [27, 14, 28, 29].

**A.1. Quaternionic holomorphic line bundles.** A *quaternionic holomorphic line bundle* over a Riemann surface  $M$  is a quaternionic line bundle  $L$  equipped with a complex structure  $J \in \Gamma(\text{End } L)$ ,  $J^2 = -\text{Id}$  and a quaternionic linear (Dirac type) differential operator  $D: \Gamma(L) \rightarrow \Gamma(\bar{K}L)$  satisfying the Leibniz rule

$$D(\psi\lambda) = (D\psi)\lambda + (\psi d\lambda)''$$

for all  $\psi \in \Gamma(L)$  and  $\lambda: M \rightarrow \mathbb{H}$ , where  $\omega'' := \frac{1}{2}(\omega + J*\omega)$  and  $\bar{K}L$  denotes the bundle of 1-forms with values in  $L$  that transform like  $*\omega = -J\omega$  for  $*$  the induced complex structure on  $T^*M$ . The degree of a quaternionic holomorphic line bundle is defined as the degree of the underlying complex line bundle  $E := \{\psi \in L \mid J\psi = \psi i\}$ . The complex structure  $J$  decomposes the operator  $D = \bar{\partial} + Q$  into  $J$ -commuting part  $\bar{\partial}$  and anti-commuting part  $Q$ . The operator  $\bar{\partial}$  respects the complex line bundle  $E$  and defines a complex holomorphic structure. The tensor field  $Q$  is called the *Hopf field* of the quaternionic holomorphic line bundle.

There are two essentially different ways how quaternionic holomorphic line bundles arise in the theory of conformal immersions of Riemann surfaces into 4-space. One of them is Möbius invariant and can be best understood within the quaternionic model of the conformal 4-sphere  $S^4 = \mathbb{HP}^1$ ; the other depends on the choice of a Euclidean subgeometry or, more precisely, on the choice of a point at infinity  $\infty \in S^4 = \mathbb{HP}^1$ .

By trivialising the bundle with a  $\bar{\partial}$ -holomorphic section one can bring the operator  $D$  to the form of a Dirac operator (1.1) acting on functions. When dealing with quaternionic holomorphic line bundles related to immersed surfaces in 3-space, such a trivialization can always be achieved globally if one uses sections and functions with  $\mathbb{Z}_2$ -monodromy which takes into account the spin structure of the immersion.

**A.2. Möbius invariant representation.** In the following we identify maps  $f: M \rightarrow S^4 = \mathbb{HP}^1$  into the conformal 4-sphere with quaternionic line subbundles  $L \subset V$  of a trivial quaternionic rank 2 bundle  $V$  over  $M$  equipped with a trivial connection.<sup>5</sup> If  $f$  is

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<sup>5</sup>Here  $V$  can be simply thought of as the Cartesian product of  $M$  with a quaternionic rank 2 vector space. The reason for preferring the language of bundles and connections is that it simplifies the treatment of immersions with Möbius monodromy, e.g. in Section 4 where we discuss minimal tori with translational periods.

a conformal immersion, the quaternionic line bundle  $V/L$  carries a unique quaternionic holomorphic structure for which all sections obtained by projection from constant sections of  $V$  are holomorphic (see the rest of the section for a more explicit coordinate description and e.g. [4] for more details). This holomorphic structure on  $V/L$  is Möbius invariant, because its definition is projectively invariant.

The conformal immersion is now encoded in the 2-dimensional linear system of holomorphic sections of  $V/L$  obtained by projection from constant sections of  $V$ . The quotient of any two linearly independent sections  $\psi_1, \psi_2$  in this linear system is a coordinate representation of the immersion in an affine chart of  $\mathbb{HP}^1$ . To see this note that the choice of  $\psi_1, \psi_2$  identifies  $V$  with the trivial  $\mathbb{H}^2$  vector bundle with trivial connection. The holomorphic sections  $\psi_1, \psi_2$  of  $V/L$  are then the projections under  $\pi: V \rightarrow V/L$  of the basis vectors  $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ . Away from the isolated points at which  $f$  goes through  $\infty = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbb{H}$ , the line bundle corresponding to the immersion can be written as  $L = \begin{pmatrix} f \\ 1 \end{pmatrix} \mathbb{H}$ , where  $f: M \rightarrow \mathbb{H} = \mathbb{HP}^1 \setminus \{\infty\}$  is the representation of the immersion in the affine chart defined by  $\infty$ . In particular, the immersion has the quotient representation

$$\psi_2 = -\psi_1 f.$$

Note that as long as the immersion does not go through  $\infty$ , the section  $\psi_1$  has no zeroes and the affine representation  $f$  is a globally smooth map  $f: M \rightarrow \mathbb{H}$ . An arbitrary holomorphic section of  $V/L$  then takes the form  $\psi_1 g$  for  $g: M \rightarrow \mathbb{H}$  a function satisfying  $*dg = N dg$ , where  $N: M \rightarrow S^2 \subset \text{Im}(\mathbb{H})$  is the so called left normal of  $f$  defined by  $*df = N df$ . In the case that  $f$  takes values in  $\mathbb{R}^3 = \text{Im}(\mathbb{H})$ , the map  $N$  is the Gauss map of the immersion.

Given an immersion into the conformal 3- or 4-sphere, a generic choice of a point  $\infty$  at infinity will avoid that the immersion goes through  $\infty$ . However, it might be preferable (for example in Section 2) to choose a point  $\infty$  at infinity for which the immersion does go through  $\infty$ . In this case, the affine representation  $f$  is smooth away from the *ends* at which it goes through  $\infty$ . At the ends the quaternion valued function  $f$  then has first order poles in the sense that  $f^{-1}$  vanishes to first order: it vanishes, because  $f$  has an end, and its vanishing order is one, because  $f$  is the affine representation of an immersion into the conformal 4-sphere.

**A.3. Euclidean Weierstrass representation.** Give a conformal immersion  $f: M \rightarrow \mathbb{R}^4 = \mathbb{H}$  of a Riemann surface into Euclidean 4-space, there are unique (up to isomorphism) quaternionic holomorphic line bundles  $L$  and  $\tilde{L}$  with holomorphic sections  $\psi$  and  $\alpha$  and a unique pairing [13, Section 2.3] between  $\tilde{L}$  and  $L$ , i.e., a quaternionic sesquilinear map  $(,): \tilde{L} \times L \rightarrow T^*M \otimes \mathbb{H}$ , such that

$$(\alpha, \psi) = df.$$

This is the quaternionic version [23] of the Weierstrass representation of  $f$  and the bundle  $\tilde{L}$  is isomorphic to  $KL^{-1}$ . Although the quaternionic holomorphic structures on the line bundles  $L$  and  $KL^{-1}$  are not Möbius invariant, the underlying paired quaternionic line bundles and their complex holomorphic structures  $\bar{\partial}$  are Möbius invariant.

If  $f$  takes values in  $\mathbb{R}^3 = \text{Im}(\mathbb{H})$ , then  $\psi \mapsto \alpha$  defines an isomorphism between  $L$  and  $\tilde{L}$ . The bundle  $L$  is then called a *quaternionic spin bundle*, the quaternionic holomorphic structure is compatible with the pairing in the sense that holomorphic sections square to closed forms, and  $f$  has Weierstrass representation  $df = (\psi, \psi)$ . In particular, the underlying complex bundle  $E$  with  $\bar{\partial}$  is then a complex holomorphic spin bundle (a square

root of the canonical bundle), because if  $\varphi$  is a  $\bar{\partial}$ -holomorphic section of  $E$ , the pairing  $(\varphi, \varphi)$  is  $j$  times a holomorphic differential.

We explain now, how the Weierstrass representation fits into the quaternionic projective picture explained above. First note that the projective differential  $\delta := \pi \nabla|_L$  of a conformal immersion  $f: M \rightarrow S^4 = \mathbb{H}\mathbb{P}^1$  is a bundle isomorphism  $\delta: L \rightarrow KV/L$ . This allows to define a Möbius invariant pairing [13, Section 2.3] between  $L$  and the bundle  $L^\perp \subset V^*$  perpendicular to  $L$  by setting

$$(\alpha, \psi) := \langle \alpha, \delta\psi \rangle = - \langle \delta^\perp \alpha, \psi \rangle,$$

where  $\delta^\perp: L^\perp \rightarrow KL^{-1} = KV^*/L^\perp$  denotes the projective derivative of  $L^\perp$ .

From now on assume that we have fixed a point  $\infty \in \mathbb{H}\mathbb{P}^1$ . Away from the ends of the immersion  $f$ , i.e., the isolated points at which  $f$  goes through  $\infty$ , the point  $\infty$  defines holomorphic structures on the bundles  $L$  and  $L^\perp$ : denote by  $\psi_1 \in H^0(V/L)$  a holomorphic section of  $V/L$  obtained by projecting a non-trivial, constant section of the quaternionic line bundle  $\infty \subset V$ . Away from the ends (which coincide with the vanishing locus of  $\psi_1$ ), there is a unique flat connection  $\nabla$  on  $V/L$  satisfying  $\nabla\psi_1 = 0$ . This connection defines holomorphic structures  $\nabla''$  on  $L^\perp = (V/L)^{-1}$  and  $d^\nabla$  on  $KV/L \cong L$ .

We show now that these holomorphic structures are the holomorphic structures occurring in the Weierstrass representation of the corresponding immersion into the Euclidean space  $\mathbb{H} = \mathbb{H}\mathbb{P}^1 \setminus \{\infty\}$ : let  $f: M \rightarrow \mathbb{H} \subset \mathbb{H}\mathbb{P}^1$  so that the corresponding quaternionic line subbundle is  $L = \psi\mathbb{H}$  with  $\psi = \begin{pmatrix} f \\ 1 \end{pmatrix}$  and  $\infty = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbb{H}$ . The section  $\psi$  is holomorphic since  $\delta\psi = \psi_1 df \in \Gamma(KV/L)$  and holomorphic sections of  $KV/L$  are precisely the sections of the form  $\psi\eta$  with  $*\eta = N\eta$  and  $d\eta = 0$ . The section  $\alpha \in \Gamma(L^\perp)$  defined by  $\langle \alpha, \psi_1 \rangle = 1$  is holomorphic as well and  $(\alpha, \psi) = df$  which is the Weierstrass representation of  $f$ .

**A.4. The spectral curve of a quaternionic holomorphic line bundle.** A conformal invariant attached to an immersion of a torus into 3-space is its *spectral curve* [27, 14, 28, 29, 5, 4]. In Appendix A.5 we define the spectral curve of an immersed torus based on the notion of spectral curve for quaternionic holomorphic line bundles of degree 0 on the torus which we discuss here. This spectral curve can be equivalently viewed as the spectral curve of a periodic 2-dimensional Dirac operator (1.1), cf. Appendix A.1.

Following [5], we define the spectral curve  $\Sigma$  of a quaternionic holomorphic line bundle  $L$  of degree 0 over a torus  $T^2 = \mathbb{C}/\Gamma$  as the Riemann surface normalizing the complex analytic set of possible Floquet multipliers (or monodromies) of non-trivial holomorphic sections of  $L$ , i.e., the set of

$$h \in \text{Hom}(\Gamma, \mathbb{C}_*) \cong \mathbb{C}_* \times \mathbb{C}_*$$

for which there exists a non-trivial solution to  $D\psi = 0$  defined on the universal covering  $\mathbb{C}$  of  $T^2 = \mathbb{C}/T^2$  that transforms according to

$$\psi(x + \gamma) = \psi(x)h_\gamma$$

for all  $x \in T^2$  and  $\gamma \in \Gamma$ . In order to justify the definition of the spectral curve  $\Sigma$  one has to verify that the possible multipliers form a 1-dimensional complex analytic set. In [5] this is proven by asymptotic analysis of a holomorphic family of elliptic operators. In addition it is shown that  $\Sigma$  has one or two ends (depending on whether its genus is infinite or finite) and one or two connected components each containing at least one end. Moreover, for a generic Floquet multiplier  $h$  that admits a non-trivial holomorphic section, the space of holomorphic sections with monodromy  $h$  is complex 1-dimensional.



**A.5. The spectral curve of a conformally immersed torus.** For an immersed torus in 4-space whose normal bundle is topologically trivial, the spectral curve can be defined using either the Möbius invariant or the Euclidean quaternionic holomorphic line bundles attached to the immersion. The following theorem shows that both possibilities lead to the same Riemann surface which is therefore Möbius invariant.<sup>6</sup>

**Theorem A.6.** *For a conformal immersion  $f: T^2 \rightarrow \mathbb{R}^4 = \mathbb{H} \subset \mathbb{HP}^1$  with topologically trivial normal bundle, the spectral curve of the Möbius invariant quaternionic holomorphic line bundle  $V/L$  coincides with the spectral curve of the Euclidean quaternionic holomorphic structure on  $L$ .*

*Proof.* Because of the asymptotics of spectral curves, i.e., the fact that they have at most two components each of which contains an end [5], it is sufficient to check that the set of possible monodromies of holomorphic sections of  $V/L$  is contained in the set of possible monodromies of holomorphic sections of  $L = KV/L$  equipped with the Euclidean holomorphic structure  $d^\nabla$ , where  $\nabla$  is the flat connection on  $V/L$  defined by the point  $\infty$ . But this immediately follows from the fact that, if  $\psi^h$  is a holomorphic section with monodromy  $h$  of  $V/L$ , by flatness of  $\nabla$  its derivative  $\nabla\psi^h$  is a holomorphic section of  $KV/L$  which obviously has the same monodromy as  $\psi^h$ .  $\square$

Note that in Theorem A.6 we assume that  $f$  does not go through the point  $\infty \in \mathbb{HP}^1$  defining the Euclidean geometry, that is, the corresponding immersion into  $\mathbb{R}^4 = \mathbb{H}$  is assumed to have no ends. In contrast to this, in the proof of Theorem 2.3 we choose  $\infty$  for which  $f$  has ends.

## APPENDIX B. CONFORMALLY IMMERSED TORI WITH REDUCIBLE SPECTRAL CURVE AND ELLIPTIC KP SOLITONS

The spectral curve  $\Sigma$  of a conformally immersed torus (Appendix A.5) or, more generally, a quaternionic holomorphic line bundle on a torus (Appendix A.4) is equipped with a pair of holomorphic maps

$$h_j: \Sigma \rightarrow \mathbb{C}_* \quad j = 1, 2$$

that describe the monodromies of holomorphic sections in the direction of a basis  $\gamma_1, \gamma_2$  of the lattice  $\Gamma$  defining the underlying torus  $T^2 = \mathbb{C}/\Gamma$ . If  $\Sigma$  has finite genus, the logarithmic derivatives

$$\eta_j = d \log(h_j) \quad j = 1, 2$$

are meromorphic 1-forms on the compactification  $\bar{\Sigma} = \Sigma \cup \{o, \infty\}$  of  $\Sigma$  with second order poles and no residues at the ends  $o, \infty$  (see e.g. Lemma 5.1 of [5]). Moreover, all periods of  $\eta_j$ ,  $j = 1, 2$  take values in  $2\pi i\mathbb{Z}$ . The existence of a pair of meromorphic forms with the given asymptotics and periodicity is a “closing condition” for the underlying Dirac potential. In integrable surface theory, a closing condition of this type probably first appeared in [17] (see Theorem 8.1 there).

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<sup>6</sup>It should be noted that when the spectral curve is defined using the Euclidean concept of Weierstrass representation as in its original definition [27] for tori in  $\mathbb{R}^3$ , its Möbius invariance is far from obvious. It was first conjectured by the second author [26] and first proven, independently, by Grinevich and Schmidt [14] and by Pinkall (unpublished). A non-normalized (more precise) version of the spectral curve is considered in [15] where it is shown that Möbius transformations of tori may result in creation and annihilation of multiple points on the corresponding spectral curve. The proof here is a variant of Pinkall’s unpublished proof.

The same kind of closing condition characterizes compact Riemann surfaces with one puncture that arise as spectral curves of elliptic KP solitons:

**Proposition B.1.** *A compact Riemann surface with one puncture is the spectral curve of an elliptic KP soliton if and only if it admits two linearly independent meromorphic 1-forms  $\eta_1, \eta_2$  with single second order poles and no residues at the puncture such that all periods are in  $2\pi i\mathbb{Z}$ .*

*Proof.* We only sketch the proof, without going into details of the underlying finite gap integration theory: a compact Riemann surface with one puncture gives rise, via the Krichever construction, to a function in  $x, y, t$  which is a meromorphic solution to the KP equation. This solution can be obtained from the Baker–Akhiezer function which is characterized as the section of a family of holomorphic line bundle on the Riemann surface. Its dynamics in  $x, y, t$  is given by linear flows in the Jacobian, realized by cocycles which linearly depend on  $x, y, t$  and describe the change of holomorphic line bundle. An elliptic KP soliton as described in [18] is precisely a finite gap KP solution that is double periodic in the  $x$ -variable. As in the case with two punctures (see e.g. p.665 in [17]), the existence of a pair of 1-forms  $\eta_1, \eta_2$  with the given asymptotics and periodicity means that there is a pair of  $x$ -values whose cocycles are coboundaries, i.e., that the dynamics in the  $x$ -direction is periodic.  $\square$

If the spectral curve of an immersed torus is reducible, the closedness condition on the full two-punctured spectral curve implies the closedness condition on each component. In particular, by the preceding proposition, each component is an elliptic KP spectral curve.

**Corollary B.2.** *If the spectral curve of a conformally immersed torus is reducible, its components are spectral curves of elliptic KP solitons.*

It would be interesting to understand whether in the special case of immersed tori in the conformal 3-sphere, reducible spectral curves are always (like in the examples discussed in Sections 3 and 4) spectral curves of elliptic KdV solitons, i.e., elliptic KP spectral curves for which the KP solitons don't depend on the  $y$ -variable and hence solve the KdV equation.

## APPENDIX C. ELLIPTIC FUNCTIONS AND LAMÉ POTENTIALS

We collect some facts about elliptic functions and Baker–Akhiezer functions for Hill's equation with 1-gap Lamé potential.

**C.1. Weierstrass's elliptic functions.** For the uniformization of a conformal 2-torus we write  $T^2 = \mathbb{C}/\{2\omega_1, 2\omega_3\}$ . The Weierstrass  $\wp$ -function is the unique elliptic function on  $T^2$  with a single pole of order two and the asymptotics  $\wp(x) = \frac{1}{x^2} + O(x^2)$ ; its other three branch points are the half lattice vectors  $\omega_1, \omega_3$  and  $\omega_2 = \omega_1 + \omega_3$  and

$$(\wp')^2 = 4(\wp(x) - e_1)(\wp(x) - e_2)(\wp(x) - e_3)$$

for  $e_j = \wp(\omega_j)$ ,  $j = 1, 2, 3$ . The Weierstrass  $\zeta$ -function is the unique function satisfying  $\zeta' = -\wp$  with  $\zeta(x) = \frac{1}{x} + O(x^3)$ . Because  $\wp$  is even,  $\zeta$  is odd and its translational periods  $\zeta(x + 2\omega_j) = \zeta(x) + 2\eta_j$  are given by  $\eta_j = \zeta(\omega_j)$ ,  $j = 1, 2, 3$ . In our notation the Legendre relation reads

$$\eta_1\omega_3 - \eta_3\omega_1 = \frac{\pi i}{2}.$$

The Weierstrass  $\sigma$ -function is the unique solution to  $\sigma' = \zeta\sigma$  with the asymptotics  $\sigma(x) = x + O(x^5)$ . Because  $\zeta$  is odd,  $\sigma$  is odd. Its monodromy is given by

$$\sigma(x + 2\omega_j) = -\sigma(x)e^{2\eta_j(x+\omega_j)}, \quad j = 1, 2, 3.$$

The entire function  $\sigma$  satisfies

$$\sigma(x) = x \prod_{(m,n) \neq 0} \left(1 - \frac{x}{\omega_{m,n}}\right) \exp\left(\frac{x}{\omega_{m,n}} + \frac{1}{2} \left(\frac{x}{\omega_{m,n}}\right)^2\right)$$

with  $\omega_{m,n} = 2m\omega_1 + 2n\omega_3$ .

**C.2. Baker function for Hill's equation with Lamé potential.** The Baker–Akhiezer function of Hill's equation

$$\Phi''(x) - 2\wp(x)\Phi(x) = E\Phi(x)$$

with Lamé potential  $-2\wp$  and spectral parameter  $E = \wp(\alpha)$  is given by

$$\Phi_\alpha(x) = \frac{\sigma(\alpha - x)}{\sigma(\alpha)\sigma(x)} e^{\zeta(\alpha)x}$$

for  $0 \neq \alpha \in \mathbb{C}/\{2\omega_1, 2\omega_3\}$ , see [18] or Chapter XXIII, 23.7 of [31] (according to which this formula goes back to Hermite 1877 and Halphen 1888). In particular, the spectral curve of Hill's operator with potential  $-2\wp(x)$  is the elliptic curve on which the potential is defined with the point 0 removed; the Weierstrass  $\wp$ -function describes the 2:1-map  $\alpha \mapsto E = \wp(\alpha)$  from the spectral curve to the spectral parameter plane and the hyperelliptic involution is  $\alpha \mapsto -\alpha$ .

The asymptotics of  $\Phi_\alpha$  is

$$\Phi_\alpha(x) = \frac{1}{x} - \frac{1}{2}\wp(\alpha)x + \frac{1}{6}\wp'(\alpha)x^2 + \dots$$

and its monodromy is given by

$$\Phi_\alpha(x + 2\omega_j) = \Phi_\alpha(x)e^{2(\zeta(\alpha)\omega_j - \alpha\eta_j)}, \quad j = 1, 2, 3.$$

As expected from the general spectral theory of Hill's equation, the Baker–Akhiezer functions  $\Phi_\alpha$  corresponding to the branch points  $\alpha = \omega_j$ ,  $j = 1, 2, 3$  of  $\wp$  have  $\mathbb{Z}_2 = \{\pm 1\}$ -monodromy (following in our case directly from the Legendre relation).

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